6^{th} IOAA ข้อ 13

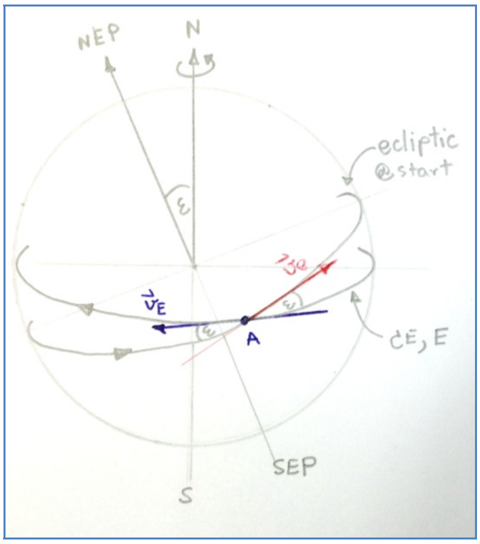
Question 13

An astronomer in the southern hemisphere contemplates the rise of the south ecliptic pole and wonders how fun it would be if the sky started spinning around the ecliptic pole, instead of the usual celestial pole.

Sketch the displacement of this observer over the Earth's surface, to observe the stars revolving around south ecliptic pole in the same direction and with the same period that they usually revolve around the south celestial pole. Sketch the observer's trajectory for one entire day. Determine its velocity (direction and speed) when crossing the Equator for the first time.

Solution

1. Considering the earth sphere and celestial sphere, we'll found that the north celestial pole(NCP) is coincide with North pole(N) and the celestial equator(CE) is parallel to the equato(E)r. The north ecliptic pole (NEP) tilted at hte angle of $\varepsilon \cong 23^o24'$ to the NCP and N. This angle also be an angle between The celestial equator and the ecliptic as shown. This picture therefore showing the surface of the earth and the sky at the same time. The position of NEP, SEP and ecliptic that was projected on the earth surface depends on the initial position and time of traveling as shown in figure 1



รูปที่ 1(Figure 1)

2. Lets suppose that the earth is not rotating and the sky is not rotating, The observer on earth will see that the NEP and SEP stay still on the sky. The question states that we want to see any object move around NEP or SEP with the period equal to the period of a circumpolar star around NCP or 1 sidereal day. Therefore he must move along the projection line of the ecliptic from west to East with the speed v_e which is

$$v_e = \frac{2\pi R}{T}$$

Where R is the Radius of the earth

Since the question stated that the trajectory started from the southern hemisphere, The first point that the trajectory cut the equator must be from the southern to the northern hemisphere. Let's say that line cross at point A

3. But since the earth are rotating, The traveler must compensate for it by moving parallel to the equator from east to west with the angular speed equal to that of the earth. The linear speed will depends on the latitude ϕ by

$$v_E = \frac{2\pi R\cos\phi}{T}$$

Please notice that on the equator $\phi = 0$ the speed of moving is $v_e = v_E = V$

4. From step 2 and 3 the traveler must move along the ecliptic when the NEP precess around NCP to the west with angular speed $\omega = \frac{2\pi}{T}$ equal to that of the earth. Therefore the angular speed of rotation and the speed of traveler on the ecliptic must be equal.

$$\begin{aligned} v_A &= V\cdot\sqrt{(\cos\varepsilon - 1)^2 + \sin^2\varepsilon} \\ &= V\cdot\sqrt{2\cdot(1-\cos\varepsilon)} \\ &\cong 0.405\cdot V \end{aligned}$$

5. Starting from point A where the traveler cross the equator for the first time, assuming that the x axis is to the east and y axis is to the north and the ecliptic making an angle of ε to the northeast of the x axis, the velocity of the traveler can be written as

$$\vec{v}_E = -\frac{2\pi R}{T}\hat{i} = -V\cdot\hat{i}$$

$$\vec{v}_e = \frac{2\pi R}{T}\left(\cos\varepsilon\cdot\hat{i} + \sin\varepsilon\cdot\hat{j}\right) = V\left(\cos\varepsilon\cdot\hat{i} + \sin\varepsilon\cdot\hat{j}\right)$$

Therefore the velocity at A is

$$\begin{aligned} \vec{v}_A &= \vec{v}_E + \vec{v}_e \\ &= V\left\{(\cos\varepsilon - 1)\hat{i} + (\sin\varepsilon)\hat{j}\right\} \end{aligned}$$

So the traveler must move at the angle $\theta = \arctan\left(\frac{\sin\varepsilon}{\cos\varepsilon - 1}\right) \cong 78^o 18'$ to the -X axis in the clockwise fashion (at an azimuth angle approximately $348^o 18'$) with the magnitude of

$$\begin{aligned} v_A &= V \cdot \sqrt{(\cos\varepsilon - 1)^2 + \sin^2\varepsilon} \\ &= V \cdot \sqrt{2 \cdot (1 - \cos\varepsilon)} \\ &\cong 0.405 \cdot V \end{aligned}$$

6. As the traveler move north the velocity used to compensate the earth rotation will become smaller as he getting to the higher latitude. Also the velocity that causes the rotation of the sky around NEP will gradually tilt down towards the equator. At the quarter distance from point A the velocity vector will be parallel to the equator and the traveler will come back to the same meridian as point A only that he will be at latitude ε, Lets called this point B . His velocity will be to the east with magnitude

$$\vec{v}_B = V \cdot (1 - \cos\varepsilon)\hat{i}$$

7. As e continued to move further his velocity vector will point to the south (because his velocity along the ecliptic will point to the south after point B). When he reach half a distance of the circle he will come back again to point A but the velocity vector will become

$$\vec{v}\,'_A = -V \cdot \left\{ (\cos\varepsilon - 1)\hat{i} + (\sin\varepsilon)\,\hat{j} \right\}$$

which means that he will move down south with an angle $\theta' = \arctan\left(\frac{\sin\varepsilon}{\cos\varepsilon - 1}\right) \cong 78^o 18'$ to the –X axis in counterclockwise fashion (at an azimuth angle of $191^o 42'$) with the magnitude of velocity of

$$v'_A = V \cdot \sqrt{2 \cdot (1 - \cos\varepsilon)} = 0.405 \cdot V$$

8. If we keep going further till the circle is complete, The trajectory will be the shape of figure 8 with the tip point at latitude $\pm\varepsilon$. The trajectory will cross the equator at point A as shoen in figure 2. The velocity at the equator is

$$\begin{aligned} v_A &= V \cdot \sqrt{(\cos\varepsilon - 1)^2 + \sin^2\varepsilon} \\ &= V \cdot \sqrt{2 \cdot (1 - \cos\varepsilon)} \\ &\cong 0.405 \cdot V \end{aligned}$$

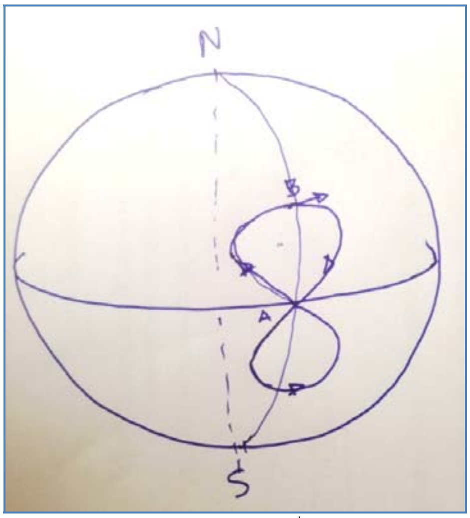


Figure 2 รูปที่ 2

Detail calculation

1. lets t be the time in the unit of sidereal hours, the angular speed on the ecliptic to the east and the angular speed to compensate the earth's rotation will be equal and equal to $\omega = \frac{2\pi}{T}$ where $T = 24^h$

2. To make the calculation easier lets assume the radius of the earth equal to 1, and the trajectory started from point A located on the +x axis of the rectangular co-ordinate centered at the centered of the earth with north pole(N) and north celestial pole (NCP) on the +Z axis. Therefore the position vector of the traveler is $\vec{r}(t=0) = \hat{i}$. The NEP will be on the YZ plane with an angle to the +Z equal to ε in the counter clockwise fashion.

3. To calculate the position of the traveler at anytime t when the traveler move along the ecliptic line *e* to the east, we using the rotational matrix to find position at time t. Lets $\lambda = \omega t$ is the angular distance along the ecliptic. The position of the traveler at ant time t can be calculate by rotating around +x axis with the angle $-\varepsilon$ with rotational matrix $\mathbf{R_x}(-\varepsilon)$ and also rotating around +z axis with angle $-\lambda$ using rotational matrix $\mathbf{R_z}(-\lambda)$ as shown

$$\begin{aligned}
\vec{r}(t) &= \mathbf{R_x}(-\varepsilon)\cdot\mathbf{R_z}(-\lambda)\cdot\vec{r}(t=0) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix}\begin{bmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\lambda \\ \cos\varepsilon\sin\lambda \\ \sin\varepsilon\sin\lambda \end{bmatrix}
\end{aligned}$$

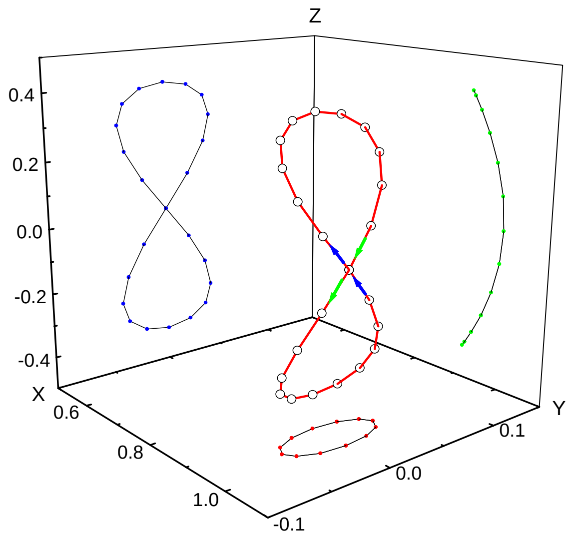
4. To compensate the earth's rotation the position vector will be rotate to the west around +z axis with angle $\theta = \omega t$ with the rotational matrix $\mathbf{R_z}(\theta)$. The results are

$$\begin{aligned}
\vec{r}'(t) &= R_z(\theta)\cdot\vec{r}(t) \\
&= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\lambda \\ \cos\varepsilon\sin\lambda \\ \sin\varepsilon\sin\lambda \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta + \sin^2\theta\cos\varepsilon \\ \sin\theta\cos\theta\cdot(\cos\varepsilon - 1) \\ \sin\theta\sin\varepsilon \end{bmatrix}
\end{aligned}$$

5. Since $\theta = \lambda = \omega t$ the position of the traveler at time t ($\vec{r}'(t)$) is shown in table

t (h)	wt	r(t)			degree	
		x	y	z	long.	lat.
0	0.0000	1.0000	0.0000	0.0000	0.00	0.00
1	0.2618	0.9945	-0.0206	0.1028	-1.18	5.90
2	0.5236	0.9794	-0.0356	0.1986	-2.08	11.45
3	0.7854	0.9589	-0.0411	0.2808	-2.46	16.31
4	1.0472	0.9383	-0.0356	0.3439	-2.17	20.12
5	1.3090	0.9233	-0.0206	0.3836	-1.28	22.56
6	1.5708	0.9178	0.0000	0.3971	0.00	23.40
7	1.8326	0.9233	0.0206	0.3836	1.28	22.56
8	2.0944	0.9383	0.0356	0.3439	2.17	20.12
9	2.3562	0.9589	0.0411	0.2808	2.46	16.31
10	2.6180	0.9794	0.0356	0.1986	2.08	11.45
11	2.8798	0.9945	0.0206	0.1028	1.18	5.90
12	3.1416	1.0000	0.0000	0.0000	0.00	0.00
13	3.4034	0.9945	-0.0206	-0.1028	-1.18	-5.90
14	3.6652	0.9794	-0.0356	-0.1986	-2.08	-11.45
15	3.9270	0.9589	-0.0411	-0.2808	-2.46	-16.31
16	4.1888	0.9383	-0.0356	-0.3439	-2.17	-20.12
17	4.4506	0.9233	-0.0206	-0.3836	-1.28	-22.56
18	4.7124	0.9178	0.0000	-0.3971	0.00	-23.40
19	4.9742	0.9233	0.0206	-0.3836	1.28	-22.56
20	5.2360	0.9383	0.0356	-0.3439	2.17	-20.12
21	5.4978	0.9589	0.0411	-0.2808	2.46	-16.31
22	5.7596	0.9794	0.0356	-0.1986	2.08	-11.45
23	6.0214	0.9945	0.0206	-0.1028	1.18	-5.90
24	6.2832	1.0000	0.0000	0.0000	0.00	0.00

6. when plot in to the graph the results are shown in figure 3



รูปที่ 3